LINEAR INTEGRALS OF A HOLONOMIC MECHANICAL SYSTEM

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The author employs the methods of tensor calculus to investigate the existence of linear integrals, and uses an example to show that a system can have a force function without possessing a linear integral [1].

1. Let the kinetic energy of a holonomic system be given by

$$T = {}^1/_2 g_{\lambda\mu} q^{\lambda\cdot} q^{\mu\cdot}$$

and the generalized forces by $Q_\varkappa$. Here and in the following the Greek indices assume the values of 1, 2, 3, ..., n, and a dot ($^\cdot$) denotes a derivative with respect to time t.

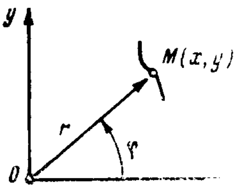

Fig. 1

Equations of motion of the system in the contravariant notation have the form

$$q^{\rho\cdot\cdot} + \Gamma^\rho_{\lambda\mu} q^{\lambda\cdot} q^{\mu\cdot} = Q^\rho \tag{1.1}$$

$$\Gamma^\rho_{\lambda\mu} = \frac{1}{2} g^{\rho\nu} \left[\frac{\partial g_{\lambda\nu}}{\partial q^\mu} + \frac{\partial g_{\nu\mu}}{\partial q^\lambda} - \frac{\partial g_{\lambda\mu}}{\partial q^\nu} \right], \qquad Q^\rho = g^{\rho\nu} Q_\nu$$

We can also write them as

$$\delta q^\rho / dt = Q^\rho \tag{1.2}$$

Let us consider the following simple example. Suppose the point M moves under the action of a central force. We shall choose the radius vector $q^1 = r$ and the polar angle $q^2 = \varphi$ as the variables. Assuming that $m = 1$, we have

$$2T = r^{2\cdot} + r^2 \varphi^{2\cdot}, \qquad Q_1 = F_2, \ Q_2 = 0$$

Let us now determine $\Gamma^\rho_{\lambda\mu}$. We find that (Fig. 1)

$$\Gamma^1_{22} = -r, \ \Gamma^2_{12} = 1/r, \ \Gamma^2_{22} = \Gamma^1_{12} = 0$$

Assume now that the system has a linear first integral

$$\lambda_\varkappa q^{\varkappa\cdot} = C, \qquad C = \text{const}$$

Differentiating it we obtain

$$\delta\lambda_\varkappa q^{\varkappa\cdot} + \lambda_\varkappa \delta q^{\varkappa\cdot} = 0$$

This and (1.2) yield

$$\delta\lambda_\varkappa / dt \, q^{\varkappa\cdot} + \lambda_\varkappa Q^\varkappa = 0$$

which together with the relation

$$\delta\lambda_\varkappa / dt = \nabla_\rho \lambda_\varkappa q^{\rho\cdot}$$

after necessary transformations gives

$$\nabla_\rho \lambda_\varkappa q^{\rho\cdot} q^{\varkappa\cdot} + \lambda_\varkappa Q^\varkappa = 0$$

From the latter relation we obtain

$$\nabla_\rho \lambda_\varkappa + \nabla_\varkappa \lambda_\rho = 0 \tag{1.3}$$

$$\lambda_\varkappa Q^\varkappa = 0 \tag{1.4}$$

Consequently condition (1.4) is both, necessary and sufficient for $\lambda_\varkappa q^\varkappa = C$ to be the first integral of the system. The first condition means that $\nabla_\varkappa \lambda_\rho$ is a skew symmetric tensor and the second one, that λ_ρ and $Q_\varkappa$ are mutually perpendicular. In addition, the first condition is independent of $Q_\varkappa$ and depends only on the form of $2T$.

In finding the linear integrals we encounter the following two problems. Firstly, we must find all covariant vectors whose covariant derivatives are skew symmetric; seondly, out of these vectors we must separate those perpendicular to the vector of generalized forces $Q_\varkappa$.

Let us return to our original example and take the vector $\lambda_1 = 0$, $\lambda_2 = r^2$. Direct computation yields

$$\nabla_1\lambda_1 = 0, \quad \nabla_2\lambda_2 = 0, \quad \nabla_1\lambda_2 = r, \quad \nabla_2\lambda_1 = -r$$

From this it follows that the first condition of (1.4) is satisfied. The second condition of (1.4) together with the relations $\lambda_1 = 0$, $\lambda_2 = r^2$, $Q^1 = F_2/r^2$ and $Q^2 = 0$ now yields

$$\lambda_\varkappa Q^\varkappa = F_2/r^2 \cdot 0 + r^2 \cdot 0 = 0$$

Using the conditions (1.4) we can now conclude that $r^2\varphi^\cdot = C$ is a linear integral.

Since φ is an ignorable coordinate, the above integral can be obtained from Lagrange's equations. We can also adopt a different approach using the rectangular coordinates $q^1 = x$ and $q^2 = y$ of the point M as parameters. The system will then have an ignorable coordinate, though possessing a linear integral

$$yx^\cdot - xy^\cdot = C$$

We return to conditions (1.4) to investigate the existence of linear integrals in the case when the system has no ignorable coordinate. Let us set $\Delta_\rho\lambda_\varkappa = \varepsilon_{\rho\varkappa}$. From (1.4) follows $\varepsilon_{\rho\varkappa} = -\varepsilon_{\varkappa\rho}$, i.e. the tensor $\varepsilon_{\rho\varkappa}$ is skew symmetric. Performing absolute differentiation we obtain

$$\nabla_\mu\nabla_\nu\lambda_\rho = \nabla_\mu\varepsilon_{\nu\rho} \tag{1.5}$$

Then we have

$$\nabla_\rho\nabla_\nu\lambda_\rho - \nabla_\nu\nabla_\rho\lambda_\rho = \nabla_\rho\varepsilon_{\nu\rho} = R_{\rho\nu\rho}{}^\mu\lambda_\mu$$

On the other hand

$$\nabla_\nu\nabla_\rho\lambda_\nu - \nabla_\rho\nabla_\nu\lambda_\nu = \nabla_\nu\varepsilon_{\rho\nu} = R_{\nu\rho\nu}{}^\mu\lambda_\mu$$

Differentiating once again we obtain

$$\nabla_\nu\nabla_{\rho}\varepsilon_{\nu\rho} = \nabla_\nu R_{\rho\nu\rho}{}^\mu\lambda_\mu + R_{\rho\nu\rho}{}^\mu\nabla_\nu\lambda_\mu$$

$$\nabla_\rho\nabla_\nu\varepsilon_{\rho\nu} = \nabla_\rho R_{\nu\rho\nu}{}^\mu\lambda_\mu + R_{\nu\rho\nu}{}^\mu\nabla_\rho\lambda_\mu$$

Adding the left-and right-hand sides respectively, we find

$$R_{\nu\rho\nu}{}^\mu\varepsilon_{\mu\rho} + R_{\nu\rho\rho}{}^\mu\varepsilon_{\nu\mu} = \nabla_\nu R_{\rho\nu\rho}{}^\mu\lambda_\mu + \nabla_\rho R_{\nu\rho\nu}{}^\mu\lambda_\mu + R_{\rho\nu\rho}{}^\mu\varepsilon_{\nu\mu} + R_{\nu\rho\nu}{}^\mu\varepsilon_{\rho\mu}$$

and finally

$$(\nabla_\nu R_{\rho\nu\rho}{}^\mu + \nabla_\rho R_{\nu\rho\nu}{}^\mu)\lambda_\mu = 2(R_{\nu\rho\nu}{}^\mu\varepsilon_{\mu\rho} + R_{\nu\rho\rho}{}^\mu\varepsilon_{\nu\mu}) \tag{1.6}$$

The number of these equations coincides with the number of the real components of the bivector $\varepsilon_{\mu\rho}$. Since the system is linear in $\varepsilon_{\mu\rho}$, we can obtain $\varepsilon_{\mu\rho}$ as a linear combination from $\lambda_\rho - \varepsilon_{\mu\rho} = \omega_{\mu\rho}{}^\varkappa\lambda_\varkappa$.

Inserting the expressions obtained into the first condition of (1.4) we arrive at

$$\nabla_\rho\lambda_\varkappa = \omega_{\rho\varkappa}{}^\nu\lambda_\nu \tag{1.7}$$

Thus the question of solvability of our problem reduces to the process of solving a system of partial differential equations. Let us construct the condition of integrability

$$\nabla_\mu\nabla_\nu\lambda_\varkappa - \nabla_\nu\nabla_\mu\lambda_\varkappa = R_{\mu\nu\varkappa}{}^\rho\lambda_\rho = \omega_{\nu\varkappa}{}^\rho\omega_{\mu\rho}{}^\pi\lambda_\pi - \omega_{\mu\varkappa}{}^\rho\omega_{\nu\rho}{}^\pi\lambda_\pi + (\nabla_\mu\omega_{\nu\varkappa}{}^\rho - \nabla_\nu\omega_{\mu\varkappa}{}^\rho)\lambda_\rho$$

The relations obtained must be fulfilled in order for the linear integral to exist and form together with the second equation of (1.4), the conditions of existence of a solution of the given type.

Let us see how the conditions (1.4) change when the system has an ignorable coordinate q^{μ_0}. In this case $\lambda_\nu = g_{\nu\mu_0}$, so that

$$\nabla_\nu\lambda_\varkappa + \nabla_\varkappa\lambda_\nu = \nabla_\nu g_{\varkappa\mu_0} + \nabla_\varkappa g_{\nu\mu_0} = \frac{\partial g_{\varkappa\mu_0}}{\partial q^\nu} - \Gamma^\rho_{\nu\varkappa}\, g_{\rho\mu_0} + \frac{\partial g_{\nu\mu_0}}{\partial q^\varkappa} - \Gamma^\rho_{\varkappa\nu}\, g_{\rho\mu_0}$$

where the differentiation over the index μ_0 is not performed. By the theorem of Ricci we have

$$\frac{\partial g_{\varkappa\mu_0}}{\partial q^\nu} - \Gamma^\rho_{\varkappa\nu}\, g_{\rho\mu_0} = \Gamma^\rho_{\nu\mu_0}\, g_{\varkappa\rho}, \qquad \frac{\partial g_{\nu\mu_0}}{\partial q^\varkappa} - \Gamma^\rho_{\varkappa\nu}\, g_{\rho\mu_0} = \Gamma^\rho_{\varkappa\mu_0}\, g_{\nu\rho}$$

in this case condition (1.3) becomes

$$\nabla_\nu\lambda_\varkappa + \nabla_\varkappa\lambda_\nu = \Gamma^\rho_{\nu\mu_0}\, g_{\varkappa\rho} + \Gamma^\rho_{\varkappa\mu_0}\, g_{\nu\rho} = \Gamma_{\varkappa,\,\nu\mu_0} + \Gamma_{\nu,\,\varkappa\mu_0} = \frac{\partial g_{\varkappa\nu}}{\partial q^{\mu_0}} = 0$$

and from this

$$\frac{\partial T}{\partial q^{\mu_0}} = \frac{1}{2}\frac{\partial g_{\varkappa\nu}}{\partial q^{\mu_0}} q^{\varkappa\cdot} q^{\nu\cdot} = 0$$

The second condition of (1.4) can be written as

$$\lambda_\varkappa Q^\varkappa = g_{\mu_0\varkappa} Q^\varkappa = Q_{\mu_0} = 0$$

Finally we find that $\partial T / \partial q^{\mu_0} = 0$ and $Q_{\mu_0} = 0$ must hold, if q^{μ_0} is an ignorable coordinate. These conditions are well known in analytical mechanics. The problem of existence of "latent" linear integrals can also be tackled in a different manner. Second equation of (1.4) upon absolute differentiation yields

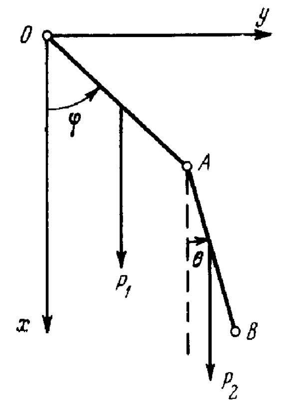

Fig. 2

$$\nabla_\rho\lambda_\varkappa Q^\varkappa + \lambda_\varkappa \nabla_\rho Q^\varkappa = 0, \qquad \nabla_\varkappa\lambda_\rho Q^\rho + \lambda_\rho \nabla_\varkappa Q^\rho = 0$$

Multiplying the first equation by Q^ρ and the second one by $Q^\varkappa$ and adding, we obtain

$$(\nabla_\varkappa\lambda_\rho + \nabla_\rho\lambda_\varkappa)\, Q^\varkappa Q^\rho + \lambda_\varkappa Q^\rho \nabla_\rho Q^\varkappa + \lambda_\rho Q^\varkappa \nabla_\varkappa Q^\rho = 2\lambda_\varkappa Q^\rho \nabla_\rho Q^\varkappa = 0 \tag{1.8}$$

We have thus obtained a different equation in $\lambda_\varkappa$. These arguments are applicable if at least one of the coefficients of $Q_\varkappa \neq 0$. The process leading to Eq. (1.8) can be applied to the equation itself repeatedly ad infinitum. If the resulting linear equations are incompatible, the system has no solution. Otherwise these equations serve only to restrict the range of variation of $\lambda_\varkappa$.

To illustrate the methods discussed we shall consider an example. A double mathematical pendulum consists of two heavy rods OA and AB, hinged cylindrically at the point A and suspended at the point O. We assume that $|OA| = |AB| = 2r$ and they are of equal mass m (Fig. 2).

Expressions for the kinetic energy, principal moments of inertia and the potential energy are, respectively,

$$2T = A\varphi^{\cdot 2} + B\theta^{\cdot 2} + 2C\cos(\theta - \varphi)\theta^\cdot\varphi^\cdot$$

$$A = \frac{16}{3} mr^2, \quad B = \frac{4}{3} mr^2, \quad C = 2mr^2, \quad U = 3mgr\cos\varphi + mgr\cos\theta$$

Setting $q^1 = \theta$ and $q^2 = \varphi$ we obtain the following expressions for Γ^k_{ij}

$$\Gamma_{11}^1 = -\Gamma_{22}^2 = \frac{C^2\sin(\theta-\varphi)\cos(\theta-\varphi)}{AB - C^2\cos^2(\theta-\varphi)};$$

$$\Gamma_{22}^1 = \frac{AC\sin(\theta-\varphi)}{AB - C^2\cos^2(\theta-\varphi)}, \qquad \Gamma_{11}^2 = \frac{-BC\sin(\theta-\varphi)}{AB - C^2\cos^2(\theta-\varphi)}$$

The remaining $\Gamma_{ij}^{k}=0$.

Let us find the components of the metric tensor $R_{\gamma\beta\mu}^{\alpha}$. As we know from [2]

$$R_{\gamma\beta\mu}^{\alpha}=\frac{\partial\Gamma_{\gamma\mu}^{\alpha}}{\partial_{q}^{\beta}}-\frac{\partial\Gamma_{\beta\mu}^{\alpha}}{\partial_{q\gamma}}+\Gamma_{\gamma\mu}^{\delta}\Gamma_{\beta\delta}^{\alpha}-\Gamma_{\gamma\delta}^{\alpha}\Gamma_{\beta\mu}^{\delta}$$

$$R_{121}{}^{2}=\frac{-BC\cos(\theta-\varphi)}{AB-C^{2}\cos^{2}(\theta-\varphi)},\qquad R_{212}{}^{1}=\frac{AC\cos(\theta-\varphi)}{AB-C^{2}\cos^{2}(\theta-\varphi)}$$

$$R_{121}{}^{1}=R_{212}{}^{2}=\frac{C^{4}\sin^{2}(\theta-\varphi)-C^{2}AB\cos^{2}(\theta-\varphi)}{(AB-C^{2}\cos^{2}(\theta-\varphi))^{2}}$$

Expression (1.6) now becomes

$$[\nabla_{1}R_{212}{}^{j}+\nabla_{2}R_{121}{}^{j}]\,\lambda_{j}=2\,[R_{121}{}^{l}\varepsilon_{l2}+R_{122}{}^{l}\varepsilon_{1l}]=2\,[R_{121}{}^{1}+R_{122}{}^{2}]\,\varepsilon_{12}=0$$

Keeping in mind that $R_{121}{}^{1}=-R_{211}{}^{2}$, we find

$$g^{jk}\,[\nabla_{1}R_{212k}+\nabla_{2}R_{121k}]\,\lambda_{j}=0$$

which can be transformed to yield finally

$$g^{jk}\,[\nabla_{1}R_{212k}+\nabla_{2}R_{121k}]=g^{j1}\nabla_{1}R_{2121}+g^{j2}\nabla\ R_{1212}$$

so that

$$g^{jk}\nabla_{k}R_{1212}\lambda_{j}=0 \tag{1.9}$$

Relations (1.4) and (1.9) represent a system of linear equations in λ_j. The condition of existance of solutions has the form

$$Q_{s}=\rho\nabla_{s}R_{1212} \tag{1.10}$$

Further

$$\nabla_{1}R_{1212}=\frac{\partial R_{1212}}{\partial q^{1}}-\Gamma_{11}{}^{1}R_{1212}-\Gamma_{11}{}^{1}R_{1212}$$

$$\nabla_{2}R_{1212}=\frac{\partial R_{1212}}{\partial q^{2}}-\Gamma_{22}{}^{2}R_{1212}-\Gamma_{22}{}^{2}R_{1212}$$

Using

$$\Gamma_{22}{}^{2}=-\Gamma_{11}{}^{1},\qquad R_{1212}=g_{p2}R_{121}{}^{p}=f(\theta-\varphi)$$

$$\frac{\partial R_{1212}}{\partial\theta}=f'(\theta-\varphi),\qquad \frac{\partial R_{1212}}{\partial\varphi}=-f'(\theta-\varphi)$$

we find

$$\nabla_{1}R_{1212}=-\nabla_{2}R_{1212} \tag{1.11}$$

Condition (1.9) in the expanded form becomes

$$g^{11}\nabla_{1}R_{1212}\lambda_{1}+g^{21}\nabla_{1}R_{1212}\lambda_{2}+g^{12}\nabla_{2}R_{1212}\lambda_{1}+g^{22}\nabla_{2}R_{1212}\lambda_{2}=0$$

Let us assume that $\nabla_{2}R_{1212}=0$, then

$$\nabla_{2}R_{1212}=g_{2p}\nabla_{2}R_{121}{}^{p}=0,\qquad \nabla_{2}R_{1211}=g_{1p}\nabla_{2}R_{121}{}^{p}=0$$

Since the determinant of this system is different from zero, we have

$$\nabla_{2}R_{121}{}^{1}=\nabla_{2}R_{121}{}^{2}=0$$

But

$$\nabla_{2}R_{121}{}^{2}=\frac{\partial R_{121}{}^{2}}{\partial\varphi}\neq0$$

The contradiction obtained shows that

$$\nabla_{2}R_{1212}\neq0$$

Simplifying we obtain

$$(g^{11}-g^{12})\,\lambda_{1}+(-g^{22}+g^{21})\,\lambda_{2}=0$$

from which we have

$$\lambda_{1}=\frac{\rho}{\Delta}\,(B+C\cos(\theta-\varphi)),\qquad \lambda_{2}=\frac{\rho}{\Delta}\,(A+C\cos(\theta-\varphi)) \tag{1.12}$$

Here
$$g_{11}=B,\ g_{22}=A,\ g_{12}=g_{21}=C\cos(\theta-\varphi)$$
$$g^{11}=g_{22}/\Delta,\quad g^{12}=g^{21}=-g_{12}/\Delta,\quad g^{22}=g_{11}/\Delta,\quad \Delta=AB-C^2\cos^2(\theta-\varphi)$$

Direct computation shows that the vector
$$\lambda_1=B+C\cos(\theta-\varphi),\ \lambda_2=A+C\cos(\theta-\varphi) \tag{1.13}$$
satisfies the condition (1.4), namely
$$\nabla_1\lambda_1=\nabla_2\lambda_2=0,\qquad \nabla_1\lambda_2+\nabla_2\lambda_1=0$$

Since all solutions of the system are given in the form (1.12), any other solution must be collinear with (1.13) so that $\mu_k=v\lambda_k$.

We shall now show that $v=$ const. Indeed
$$\nabla_1\mu_1=\nabla_1\mathbf{v}\lambda_1+\mathbf{v}\cdot\nabla_1\lambda_1=\nabla_1\mathbf{v}\cdot\lambda_1=0$$
$$\nabla_2\mu_2=\nabla_2\mathbf{v}\lambda_2+\mathbf{v}\cdot\nabla_2\lambda_2=\nabla_2\mathbf{v}\lambda_2=0 \tag{1.14}$$
$$\nabla_1\mu_2+\nabla_2\mu_1=\nabla_1\mathbf{v}\lambda_2+\mathbf{v}\nabla_1\lambda_2+\nabla_2\mathbf{v}\lambda_1+\mathbf{v}\nabla_2\lambda_1=\nabla_1v\lambda_2+\nabla_2v\lambda_1=0$$

The following cases are possible:

1) let $\lambda_1\neq 0,\ \lambda_2\neq 0$; then
$$\nabla_1\mathbf{v}=\partial\mathbf{v}/\partial\theta=0,\qquad \nabla_2\mathbf{v}=\partial\mathbf{v}/\partial\varphi=0$$
consequently $\mathbf{v}=$ const.

2) let $\lambda_1=0,\ \lambda_2\neq 0$ or vice versa; then
$$\nabla_2\mathbf{v}=0,\qquad \nabla_1\mathbf{v}\lambda_2=0$$
i.e. $\Delta_1\mathbf{v}=0$, which brings us back to the case (1).

3) the case $\lambda_1=\lambda_2=0$ is of no interest.

We have thus shown that $v=$ const.

Other solutions of Eq.(1 3) exist appart from (1.13), but they can all be obtained by multiplying the latter solution by an arbitrary constant.

To investigate the existence of a linear integral we shall turn to condition (1.10)
$$Q_1=\rho\nabla_1R_{1212},\qquad Q_2=\rho\nabla_2R_{1212}=-\rho\nabla_1R_{1212}$$

Therefore $Q_1=-Q_2$ must hold. Keeping in mind the expression for the kinetic energy we obtain
$$Q_1=\frac{\partial U}{\partial\theta}=-mgr\sin\theta,\quad Q_2=\frac{\partial U}{\partial\varphi}=-3mgr\sin\varphi,\quad Q_1\neq Q_2$$

Thus the system has no linear integral, although it has a force function.

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